

Some refined higher type adjunction inequalities on 4-manifolds

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Abstract.

We further sharpen higher type adjunction inequalities of P. Ozsváth and Z. Szabó on a 4-manifold M with a nonzero Seiberg-Witten invariant for a Spin^c structure \mathfrak{s} , when an embedded surface $\Sigma \subset M$ satisfies $[\Sigma] \cdot [\Sigma] \geq 0$ and

$$|\langle [\Sigma], c_1(\mathfrak{s}) \rangle| + [\Sigma] \cdot [\Sigma] \geq 2b_1(M).$$

Keywords: Seiberg-Witten equations, adjunction inequality

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1. INTRODUCTION

Given a Spin^c structure \mathfrak{s} on a smooth closed oriented Riemannian 4-manifold M , for a section Φ of the plus spinor bundle W_+ of \mathfrak{s} and a $u(1)$ connection A on $\det W_+$, the Seiberg-Witten equations are given by

$$\begin{cases} D_A \Phi = 0 \\ F_A^+ + i\eta = \Phi \otimes \Phi^* - \frac{|\Phi|^2}{2} \text{Id}, \end{cases}$$

where D_A and F_A^+ respectively denote the associated Dirac operator and the self-dual part of the curvature dA of A , a self-dual 2-form η is a generic perturbation term, and lastly the identification of both sides in the second equation comes from the Clifford action.

Its moduli space \mathfrak{M} , i.e. the space of solutions modulo bundle automorphisms known as the gauge group $\mathcal{G} := \text{Map}(M, S^1)$ is a smooth orientable manifold of dimension

$$d(\mathfrak{s}) := \frac{c_1(\mathfrak{s})^2 - (2\chi(M) + 3\tau(M))}{4},$$

where χ and τ denote Euler characteristic and signature respectively.

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The intersection theory on \mathfrak{M} produces Seiberg-Witten invariants in the form of a function

$$\begin{aligned} SW_{M,\mathfrak{s}} &: \mathbb{A}(M) \rightarrow \mathbb{Z} \\ \alpha &\longmapsto \langle \mu(\alpha), [\mathfrak{M}] \rangle \end{aligned}$$

where $\mathbb{A}(M)$ denotes the graded algebra obtained by tensoring the exterior algebra on $H_1(M; \mathbb{Z})$ with grading one and the polynomial algebra on $H_0(M; \mathbb{Z})$ with grading two, and the algebra homomorphism μ is defined as follows. For the positive generator U of $H_0(M; \mathbb{Z})$, $\mu(U)$ is the first Chern class of a principal S^1 bundle \mathfrak{M}_o over \mathfrak{M} , where \mathfrak{M}_o is the solution space modulo the based gauge group $\mathcal{G}_o = \{h \in \mathcal{G} | h(o) = 1\}$ for a fixed base point $o \in M$. For $[c] \in H_1(M, \mathbb{Z})$,

$$\mu([c]) = Hol_c^*([d\theta])$$

where $[d\theta]$ is the positive generator of $H^1(S^1, \mathbb{Z})$, and $Hol_c : \mathfrak{M} \rightarrow S^1$ is given by the holonomy of each connection around c .

Although $SW_{M,\mathfrak{s}}$ is a diffeomorphism invariant of M for $b_2^+(M) > 1$, when $b_2^+(M) = 1$, it depends on a chamber which is a connected component of

$$\{\omega \in H^2(M; \mathbb{R}) - 0 \mid \omega^2 \geq 0\}$$

so that Seiberg-Witten invariants may change according to which chamber the self-dual harmonic part of $-2\pi c_1(\mathfrak{s}) + \eta$ belongs to.

Definition 1. We call a $Spin^c$ structure \mathfrak{s} with $SW_{M,\mathfrak{s}}(U^{\frac{d(\mathfrak{s})}{2}}) \neq 0$ a basic class of M , and M is called of simple type, if $d(\mathfrak{s}) = 0$ for any basic class \mathfrak{s} of M .

One of major applications of Seiberg-Witten theory is the resolution of the (generalized) Thom conjecture stating that a closed symplectic surface in a closed symplectic 4-manifold is genus-minimizing in its homology class. This is generalized to the adjunction inequality on any 4-manifold with a nontrivial Seiberg-Witten invariant. On a smooth closed oriented 4-manifold M of $b_2^+(M) > 1$ and simple type, any embedded closed surface Σ with genus $g(\Sigma) > 0$ satisfies

$$| \langle [\Sigma], c_1(\mathfrak{s}) \rangle | + [\Sigma] \cdot [\Sigma] \leq 2g(\Sigma) - 2$$

for any basic class \mathfrak{s} . If $[\Sigma] \cdot [\Sigma] \geq 0$, the simply type condition is unnecessary and moreover the inequality can be enhanced to the following.

Theorem 1.1 (P. Ozsváth and Z. Szabó [4]). *Let M be a smooth closed oriented 4-manifold and $\Sigma \subset M$ be an embedded oriented surface with genus $g(\Sigma) > 0$ representing a non-torsion homology class with $[\Sigma] \cdot [\Sigma] \geq 0$.*

If $b_2^+(M) > 1$, then

$$|\langle [\Sigma], c_1(\mathfrak{s}) \rangle| + [\Sigma] \cdot [\Sigma] + (2 - \min(b_1(M), 1))d(\mathfrak{s}) \leq 2g(\Sigma) - 2$$

for each basic class \mathfrak{s} . If $b_2^+(M) = 1$, then for each basic class \mathfrak{s} with

$$-\langle [\Sigma], c_1(\mathfrak{s}) \rangle + [\Sigma] \cdot [\Sigma] \geq 0,$$

$$-\langle [\Sigma], c_1(\mathfrak{s}) \rangle + [\Sigma] \cdot [\Sigma] + (2 - \min(b_1(M), 1))d(\mathfrak{s}) \leq 2g(\Sigma) - 2,$$

where the Seiberg-Witten invariant is calculated in the chamber containing $PD[\Sigma]$.

To state a more general version of Theorem 1.1, recall that the first homology of a closed oriented surface can be viewed as a symplectic vector space given by the intersection pairing, and a basis for a symplectic vector space is called *symplectic* if the symplectic form takes the standard form in the basis. We define an invariant for an (embedded) oriented surface in a 4-manifold, which is a crucial tool in the present paper.

Definition 2. Let M be a 4-manifold. For a closed oriented surface Σ with genus $g > 0$ embedded in M , define $l(\Sigma)$ to be the maximum of integers l so that there is a symplectic basis $\{A_j, B_j\}_{j=1}^g$ in $H_1(\Sigma; \mathbb{Z})$ satisfying that $i_*(A_j) = 0$ in $H_1(M; \mathbb{Q})$ for $j = 1, \dots, l$, where $i : \Sigma \rightarrow M$ is the inclusion map.

Theorem 1.2 (P. Ozsváth and Z. Szabó [4]). Let M be a smooth closed oriented 4-manifold of $b_2^+(M) > 0$ and $\Sigma \subset M$ be an embedded oriented surface with genus $g(\Sigma) > 0$ representing a non-torsion homology class with $\Sigma \cdot \Sigma \geq 0$.

Let $a \in \mathbb{A}(M)$ and $b \in \mathbb{A}(\Sigma)$ with degree $d(b) \leq l(\Sigma)$, and suppose \mathfrak{s} is a $Spin^c$ structure with $SW_{M, \mathfrak{s}}(a \cdot i_*(b)) \neq 0$ (in the chamber containing $PD[\Sigma]$, if $b_2^+(M) = 1$).

If $b_2^+(M) > 1$, then

$$|\langle [\Sigma], c_1(\mathfrak{s}) \rangle| + [\Sigma] \cdot [\Sigma] + 2d(b) \leq 2g(\Sigma) - 2.$$

If $b_2^+(M) = 1$ and

$$-\langle [\Sigma], c_1(\mathfrak{s}) \rangle + [\Sigma] \cdot [\Sigma] \geq 0,$$

then

$$-\langle [\Sigma], c_1(\mathfrak{s}) \rangle + [\Sigma] \cdot [\Sigma] + 2d(b) \leq 2g(\Sigma) - 2.$$

Furthermore for b with $d(b) > l(\Sigma)$, the similar inequalities hold with $2d(b)$ replaced with $d(b)$.

Here the inclusion map $i : \Sigma \rightarrow M$ induces a map $i_* : \mathbb{A}(\Sigma) \rightarrow \mathbb{A}(M)$ for likewise defined $\mathbb{A}(\Sigma)$. From now on, i_* will denote the homomorphism both on homologies and $\mathbb{A}(\cdot)$ induced by imbedding, and $d(b)$ will denote

the degree of $b \in \mathbb{A}(\Sigma)$. We improve the above theorem by replacing the condition involving $l(\Sigma)$ with a condition on $[\Sigma]$.

Theorem 1.3. *Let M be a smooth closed oriented 4-manifold of $b_2^+(M) > 0$ and $\Sigma \subset M$ be an embedded oriented surface with genus $g(\Sigma) > 0$ representing a non-torsion homology class with $[\Sigma] \cdot [\Sigma] \geq 0$.*

Let $a \in \mathbb{A}(M)$ and $b \in \mathbb{A}(\Sigma)$, and suppose \mathfrak{s} is a Spin^c structure with $SW_{M,\mathfrak{s}}(a \cdot i_(b)) \neq 0$ (in the chamber containing $PD[\Sigma]$, if $b_2^+(M) = 1$).*

If $b_2^+(M) > 1$ and

$$|\langle [\Sigma], c_1(\mathfrak{s}) \rangle| + [\Sigma] \cdot [\Sigma] \geq 2b_1(M),$$

then

$$|\langle [\Sigma], c_1(\mathfrak{s}) \rangle| + [\Sigma] \cdot [\Sigma] + 2d(b) \leq 2g(\Sigma) - 2.$$

If $b_2^+(M) = 1$ and

$$-\langle [\Sigma], c_1(\mathfrak{s}) \rangle + [\Sigma] \cdot [\Sigma] \geq 2b_1(M),$$

then

$$-\langle [\Sigma], c_1(\mathfrak{s}) \rangle + [\Sigma] \cdot [\Sigma] + 2d(b) \leq 2g(\Sigma) - 2.$$

In case that $a = 1$ and $b = U^{\frac{d(\mathfrak{s})}{2}}$ where U also denotes the positive generator of $H_0(\Sigma; \mathbb{Z})$ from now on, this theorem generalizes Theorem 1.1 and it can be further extended to the following.

Theorem 1.4. *Let M be a smooth closed oriented 4-manifold of $b_2^+(M) > 0$ and $\Sigma \subset M$ be an embedded oriented surface with genus $g(\Sigma) > 0$ representing a non-torsion homology class with $[\Sigma] \cdot [\Sigma] \geq 0$. Suppose \mathfrak{s} is a basic class (in the chamber containing $PD[\Sigma]$, if $b_2^+(M) = 1$).*

When $b_2^+(M) > 1$, if

$$(1.1) \quad |\langle [\Sigma], c_1(\mathfrak{s}) \rangle| + 3[\Sigma] \cdot [\Sigma] \geq 2b_1(M),$$

then

$$|\langle [\Sigma], c_1(\mathfrak{s}) \rangle| + [\Sigma] \cdot [\Sigma] + 2d(\mathfrak{s}) - 2b_1(M) \leq 2g(\Sigma) - 2.$$

When $b_2^+(M) = 1$, if

$$(1.2) \quad -\langle [\Sigma], c_1(\mathfrak{s}) \rangle + 3[\Sigma] \cdot [\Sigma] \geq 2b_1(M),$$

and

$$(1.3) \quad -\langle [\Sigma], c_1(\mathfrak{s}) \rangle + [\Sigma] \cdot [\Sigma] \geq 0,$$

then

$$-\langle [\Sigma], c_1(\mathfrak{s}) \rangle + [\Sigma] \cdot [\Sigma] + 2d(\mathfrak{s}) - 2b_1(M) \leq 2g(\Sigma) - 2.$$

We remark that Theorem 1.4 improves Theorem 1.1 only when $b_1(M) > 0$ and $d(\mathfrak{s}) > 2$.

2. SOME ALGEBRAIC LEMMAS

Lemma 2.1. *Let V be a symplectic vector space of dimension $2g \geq 4$ with a symplectic basis $\{A_j, B_j\}_{j=1}^g$. Then for any integers r and s , $\{A'_j, B'_j\}_{j=1}^g$ where*

$$A'_1 = A_1 - rA_3, \quad A'_2 = A_2 - sA_3, \quad B'_3 = B_3 + rB_1 + sB_2,$$

and other A'_j and B'_j are the same as A_j and B_j respectively is also a symplectic basis of V .

Proof. One can check it by a simple computation. \square

Also note that since the above basis change is given by an integral symplectic matrix, its inverse is also an integral symplectic matrix. The following is our key lemma.

Lemma 2.2. *Let $i : F \rightarrow M$ be an embedding of a closed oriented surface F with genus $g > 0$ into a 4-manifold M . Then any symplectic basis $\{A_j, B_j\}_{j=1}^g$ in $H_1(F; \mathbb{Z})$ such that*

$$i_*(A_1) = \cdots = i_*(A_{l(F)}) = 0$$

in $H_1(M; \mathbb{Q})$ satisfies that the kernel of

$$i_* : \mathbb{Q}\langle A_1, \dots, A_g \rangle \rightarrow H_1(M; \mathbb{Q})$$

has dimension $l(F)$, where $\mathbb{Q}\langle A_1, \dots, A_g \rangle$ is the g -dimensional \mathbb{Q} -vector subspace of $H_1(F; \mathbb{Q})$ generated by A_1, \dots, A_g .

Proof. Let $\{A_j, B_j\}_{j=1}^g$ be a symplectic basis in $H_1(F; \mathbb{Z})$, which realizes $l(F)$, i.e. $i_*(A_1) = \cdots = i_*(A_{l(F)}) = 0$ in $H_1(M; \mathbb{Q})$.

When $g = 1$, if $i_*(cA_1)$ for $c \in \mathbb{Q} - \{0\}$ is zero in $H_1(M; \mathbb{Q})$, then so is $i_*(A_1)$, and hence $\dim \ker i_* = l(F)$.

When $g \geq 2$, assume to the contrary that there exists a nonzero vector $v \in \mathbb{Q}\langle A_1, \dots, A_g \rangle$ generated by $A_{l(F)+1}, \dots, A_g$ such that $i_*(v) = 0$ in $H_1(M; \mathbb{Q})$. By multiplying a rational number, if necessary, we may let

$$v = \sum_{j=l(F)+1}^g a_j A_j$$

where a_j 's are integers such that their greatest common divisor is 1.

Now let's call the number of nonzero a_j 's N . The $N = 1$ case is immediately excluded, because it is a contradiction to the definition of $l(F)$ as the maximum of l 's.

In the $N = 2$ case, let's say $v = a_m A_m + a_n A_n$ for $\gcd(a_m, a_n) = 1$. Take integers p and q such that $pa_m + qa_n = 1$, and we modify the above

symplectic basis by replacing A_m, B_m, A_n, B_n with

$$\begin{aligned} A'_m &= v, & B'_m &= pB_m + qB_n, \\ A'_n &= pA_n - qA_m, & B'_n &= a_mB_n - a_nB_m \end{aligned}$$

respectively. One can easily check this new basis is still symplectic. But the fact that $i_*(A'_m)$ is zero in $H_1(M; \mathbb{Q})$ along with $i_*(A_j)$ for $j = 1, \dots, l(F)$ is again contradictory to the definition of $l(F)$.

For the higher N cases, we will use induction on N . Suppose that $N \leq k$ cases lead to contradictions, and we need to prove for the $N = k + 1$ case. Let's re-denote those nonzero a_j 's by $a_m, a_{m+1}, \dots, a_{m+k}$ such that $\gcd(a_m, a_{m+1})$ has the smallest value among all possible $\gcd(a_i, a_j)$ for $i \neq j$. There exist integers r and s such that $a'_{m+2} := a_{m+2} + ra_m + sa_{m+1}$ satisfies

$$(2.4) \quad 0 \leq a'_{m+2} \leq \gcd(a_m, a_{m+1}) - 1.$$

Then we modify the symplectic basis by replacing A_m, A_{m+1}, B_{m+2} with

$$\begin{aligned} A'_m &= A_m - rA_{m+2}, & A'_{m+1} &= A_{m+1} - sA_{m+2}, \\ B'_{m+2} &= B_{m+2} + rB_m + sB_{m+1} \end{aligned}$$

respectively. By Lemma 2.1, this new basis is symplectic, and v can be expressed as

$$v = a_mA'_m + a_{m+1}A'_{m+1} + a'_{m+2}A_{m+2} + \dots$$

where \dots terms are the same as before. If $a'_{m+2} = 0$, then it is reduced to the $N = k$ case, and otherwise we keep doing this process finite times until we make certain a_j zero, because (2.4) implies that $\min_{i \neq j} \gcd(a_i, a_j)$ is reduced at least by 1 whenever performing this symplectic basis change. This completes the proof. \square

The above lemma can be rephrased as the following.

Lemma 2.3. *Under the assumptions of Lemma 2.2,*

$$g - b_1(M) \leq l(F).$$

Proof. For a symplectic basis in $H_1(F; \mathbb{Z})$ realizing $l(F)$, Lemma 2.2 dictates that the homomorphism $i_* : \mathbb{Q}\langle A_1, \dots, A_g \rangle \rightarrow H_1(M; \mathbb{Q})$ satisfies

$$g - l(F) = \dim(\text{Im}(i_*)) \leq b_1(M).$$

\square

Theorem 2.4. *Let M be a smooth closed oriented 4-manifold of $b_2^+(M) > 0$ and $\Sigma \subset M$ be an embedded oriented surface with genus $g(\Sigma) > 0$ representing a non-torsion homology class with $[\Sigma] \cdot [\Sigma] \geq 0$.*

Let $a \in \mathbb{A}(M)$ and $b \in \mathbb{A}(\Sigma)$, and suppose \mathfrak{s} is a Spin^c structure with $SW_{M, \mathfrak{s}}(a \cdot i_(b)) \neq 0$ (in the chamber containing $PD[\Sigma]$, if $b_2^+(M) = 1$).*

Suppose that

$$|\langle [\Sigma], c_1(\mathfrak{s}) \rangle| + [\Sigma] \cdot [\Sigma] \geq 2b_1(M),$$

when $b_2^+(M) > 1$, and

$$-\langle [\Sigma], c_1(\mathfrak{s}) \rangle + [\Sigma] \cdot [\Sigma] \geq 2b_1(M),$$

when $b_2^+(M) = 1$.

Then

$$d(b) \leq g(\Sigma) - b_1(M).$$

Proof. Let's first consider the $b_2^+(M) > 1$ case. Assume to the contrary that $d(b) > g(\Sigma) - b_1(M)$. Let $\Sigma' \subset M$ be an embedded oriented surface obtained by adding $d(b) - g(\Sigma) + b_1(M)$ topologically trivial handles to Σ so that $[\Sigma'] = [\Sigma]$, and $g(\Sigma') = d(b) + b_1(M)$. Moreover $\mathbb{A}(\Sigma)$ naturally injects into $\mathbb{A}(\Sigma')$, and

$$d(b) = g(\Sigma') - b_1(M) \leq l(\Sigma')$$

by Lemma 2.3.

Now we have

$$\begin{aligned} |\langle [\Sigma'], c_1(\mathfrak{s}) \rangle| + [\Sigma'] \cdot [\Sigma'] + 2d(b) &= |\langle [\Sigma], c_1(\mathfrak{s}) \rangle| + [\Sigma] \cdot [\Sigma] + 2d(b) \\ &\geq 2b_1(M) + 2d(b) \\ &= 2g(\Sigma') \\ &> 2g(\Sigma') - 2, \end{aligned}$$

which is a contradiction to Theorem 1.2 applied to Σ' . Therefore

$$d(b) \leq g(\Sigma) - b_1(M).$$

The case $b_2^+(M) = 1$ can be proved in the same way as above by replacing $|\cdot|$ with a minus sign in $|\langle [\Sigma], c_1(\mathfrak{s}) \rangle|$ and $|\langle [\Sigma'], c_1(\mathfrak{s}) \rangle|$. \square

Remark Note that $|\langle [\Sigma], c_1(\mathfrak{s}) \rangle| + [\Sigma] \cdot [\Sigma]$ is even for any Spin^c structure \mathfrak{s} and any closed surface Σ by the Wu formula. \square

3. PROOF OF THEOREM 1.3

By Lemma 2.3 and Theorem 2.4, we have

$$d(b) \leq g(\Sigma) - b_1(M) \leq l(\Sigma).$$

Then the application of Theorem 1.2 gives the desired result.

4. PROOF OF THEOREM 1.4

Lemma 4.1. *If an additional condition $[\Sigma] \cdot [\Sigma] \leq \min(b_1(M), \frac{d(\mathfrak{s})}{2})$ is satisfied, then the desired adjunction inequalities hold.*

Proof. We use the blow-up technique as in [2, 3, 4]. Take $\hat{M} = M \# r \overline{\mathbb{C}P}_2$ for $r \geq 0$ and let $\hat{\Sigma}$ be the “proper transform” of Σ so that

$$[\hat{\Sigma}] = [\Sigma] - E_1 - \cdots - E_r,$$

where E_i ’s are the classes of exceptional spheres.

Let $\hat{\mathfrak{s}}$ be the Spin^c structure on \hat{M} which agrees with \mathfrak{s} in the complement of exceptional spheres and has 1st Chern class

$$c_1(\hat{\mathfrak{s}}) = c_1(\mathfrak{s}) - 3 \sum_{i=1}^r \text{PD}[E_i].$$

By simple computations,

$$[\hat{\Sigma}] \cdot [\hat{\Sigma}] = [\Sigma] \cdot [\Sigma] - r,$$

and

$$d(\hat{\mathfrak{s}}) = d(\mathfrak{s}) - 2r.$$

First, let’s prove for the $b_2^+(M) > 1$ case. Without loss of generality we may assume $\langle [\Sigma], c_1(\mathfrak{s}) \rangle \leq 0$ by replacing $[\Sigma]$ with $-[\Sigma]$ if necessary. Then

$$|\langle [\hat{\Sigma}], c_1(\hat{\mathfrak{s}}) \rangle| = |\langle [\Sigma], c_1(\mathfrak{s}) \rangle| + 3r,$$

and

(4.1)

$$|\langle [\hat{\Sigma}], c_1(\hat{\mathfrak{s}}) \rangle| + [\hat{\Sigma}] \cdot [\hat{\Sigma}] + 2d(\hat{\mathfrak{s}}) = |\langle [\Sigma], c_1(\mathfrak{s}) \rangle| + [\Sigma] \cdot [\Sigma] + 2d(\mathfrak{s}) - 2r.$$

If we take $r = [\Sigma] \cdot [\Sigma]$, then $[\hat{\Sigma}] \cdot [\hat{\Sigma}] \geq 0$, $d(\hat{\mathfrak{s}}) \geq 0$, and

$$\begin{aligned} |\langle [\hat{\Sigma}], c_1(\hat{\mathfrak{s}}) \rangle| + [\hat{\Sigma}] \cdot [\hat{\Sigma}] &= |\langle [\Sigma], c_1(\mathfrak{s}) \rangle| + 3[\Sigma] \cdot [\Sigma] \\ &\geq 2b_1(M). \end{aligned}$$

By the well-known blow-up formula [1, 5] of Seiberg-Witten invariants,

$$SW_{\hat{M}, \hat{\mathfrak{s}}}(U^{\frac{d(\hat{\mathfrak{s}})}{2}}) = SW_{M, \mathfrak{s}}(U^{\frac{d(\mathfrak{s})}{2}}) \neq 0,$$

and hence $\hat{\mathfrak{s}}$ is a basic class. We can now apply Theorem 1.3 to \hat{M} with $a = 1$ and $b = U^{\frac{d(\hat{\mathfrak{s}})}{2}}$ to obtain

$$|\langle [\hat{\Sigma}], c_1(\hat{\mathfrak{s}}) \rangle| + [\hat{\Sigma}] \cdot [\hat{\Sigma}] + 2d(\hat{\mathfrak{s}}) \leq 2g(\hat{\Sigma}) - 2 = 2g(\Sigma) - 2.$$

Combining this with (4.1) and the assumption $b_1(M) \geq [\Sigma] \cdot [\Sigma] = r$, we get the desired adjunction inequality.

The proof for the $b_2^+(M) = 1$ case proceeds in the same way as above by replacing $|\cdot|$ with a minus sign in $|\langle[\hat{\Sigma}], c_1(\hat{\mathfrak{s}})\rangle|$ and $|\langle[\Sigma], c_1(\mathfrak{s})\rangle|$. In this case, the blow-up formula says that $SW_{M,\mathfrak{s}}(U^{\frac{d(\mathfrak{s})}{2}})$ calculated in the chamber containing $\text{PD}[\Sigma]$ is equal to $SW_{\hat{M},\hat{\mathfrak{s}}}(U^{\frac{d(\hat{\mathfrak{s}})}{2}})$ calculated in the chamber containing $\text{PD}[\hat{\Sigma}]$, which is what we need for the application of Theorem 1.3. \square

Lemma 4.2. *If an additional condition $b_1(M) \leq \min([\Sigma] \cdot [\Sigma], \frac{d(\mathfrak{s})}{2})$ is satisfied, then the desired adjunction inequalities hold.*

Proof. For this lemma, we need neither (1.1) nor (1.2). The proof is similar to the previous lemma, and we adopt the same notation. Here we will take r to be $b_1(M)$.

First let's consider the $b_2^+(M) > 1$ case, and without loss of generality we may assume $\langle[\Sigma], c_1(\mathfrak{s})\rangle \leq 0$ by replacing $[\Sigma]$ with $-[\Sigma]$ if necessary. By the assumption, we still have that $[\hat{\Sigma}] \cdot [\hat{\Sigma}] \geq 0$, $d(\hat{\mathfrak{s}}) \geq 0$, and

$$\begin{aligned} |\langle[\hat{\Sigma}], c_1(\hat{\mathfrak{s}})\rangle| + [\hat{\Sigma}] \cdot [\hat{\Sigma}] &= |\langle[\Sigma], c_1(\mathfrak{s})\rangle| + 2r + [\Sigma] \cdot [\Sigma] \\ &\geq 2b_1(M). \end{aligned}$$

Again by the blow-up formula [1, 5], $\hat{\mathfrak{s}}$ is a basic class, and hence Theorem 1.3 applied to \hat{M} with $a = 1$ and $b = U^{\frac{d(\hat{\mathfrak{s}})}{2}}$ gives

$$|\langle[\hat{\Sigma}], c_1(\hat{\mathfrak{s}})\rangle| + [\hat{\Sigma}] \cdot [\hat{\Sigma}] + 2d(\hat{\mathfrak{s}}) \leq 2g(\hat{\Sigma}) - 2 = 2g(\Sigma) - 2.$$

Combining this with (4.1), we get the desired adjunction inequality.

Again when $b_2^+(M) = 1$, the proof goes through in the same way as the $b_2^+(M) > 1$ case by replacing $|\cdot|$ with a minus sign in $|\langle[\hat{\Sigma}], c_1(\hat{\mathfrak{s}})\rangle|$ and $|\langle[\Sigma], c_1(\mathfrak{s})\rangle|$. \square

We divide the proof of Theorem 1.4 into two cases according to whether $b_1(M) \leq \frac{d(\mathfrak{s})}{2}$ or not. Suppose the first case. If $b_1(M) \geq [\Sigma] \cdot [\Sigma]$, then the proof is done by Lemma 4.1, and if $b_1(M) \leq [\Sigma] \cdot [\Sigma]$, then the proof is given by Lemma 4.2.

Now suppose $b_1(M) > \frac{d(\mathfrak{s})}{2}$. Then

$$|\langle[\Sigma], c_1(\mathfrak{s})\rangle| + [\Sigma] \cdot [\Sigma] + 2d(\mathfrak{s}) - 2b_1(M) < |\langle[\Sigma], c_1(\mathfrak{s})\rangle| + [\Sigma] \cdot [\Sigma] + d(\mathfrak{s}),$$

and hence when $b_2^+(M) > 1$, the RHS is less than or equal to $2g(\Sigma) - 2$ by Theorem 1.1. If $b_2^+(M) = 1$, we can also apply Theorem 1.1 due to the condition (1.3), and hence we deduce that

$$\begin{aligned} -\langle[\Sigma], c_1(\mathfrak{s})\rangle + [\Sigma] \cdot [\Sigma] + 2d(\mathfrak{s}) - 2b_1(M) &< -\langle[\Sigma], c_1(\mathfrak{s})\rangle + [\Sigma] \cdot [\Sigma] + d(\mathfrak{s}) \\ &\leq 2g(\Sigma) - 2. \end{aligned}$$

This completes the proof.

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